

A simple maximality principle

Joel David Hamkins*

THE CITY UNIVERSITY OF NEW YORK[†]

CARNEGIE MELLON UNIVERSITY[‡]

<http://math.gc.cuny.edu/faculty/hamkins>

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Abstract. In this paper, following an idea of Christophe Chalons, I propose a new kind of forcing axiom, the *Maximality Principle*, which asserts that any sentence φ holding in some forcing extension $V^{\mathbb{P}}$ and all subsequent extensions $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ holds already in V . It follows, in fact, that such sentences must also hold in all forcing extensions of V . In modal terms, therefore, the Maximality Principle is expressed by the scheme $(\Diamond\Box\varphi) \implies \Box\varphi$, and is equivalent to the modal theory $S5$. In this article, I prove that the Maximality Principle is relatively consistent with ZFC. A boldface version of the Maximality Principle, obtained by allowing real parameters to appear in φ , is equiconsistent with the scheme asserting that $V_\delta \prec V$ for an inaccessible cardinal δ , which in turn is equiconsistent with the scheme asserting that ORD is Mahlo. The strongest principle along these lines is $\Box\mathbb{MP}$, which asserts that \mathbb{MP} holds in V and all forcing extensions. From this, it follows that $0^\#$ exists, that $x^\#$ exists for every set x , that projective truth is invariant by forcing, that Woodin cardinals are consistent and much more. Many open questions remain.

1 The Maximality Principle

Christophe Chalons has introduced a delightful new axiom, asserting in plain language that anything that is forceable and not subsequently unforceable is true. Specifically, in [2] he proposes the following principle:

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The Chalons Maximality Principle Suppose φ is a sentence and for every inaccessible cardinal κ there is a forcing notion $\mathbb{P} \in V_\kappa$ such that $V_\kappa^\mathbb{P} \models \varphi$ and for any such \mathbb{P} and any further forcing $\dot{\mathbb{Q}}$ it holds that $V_\kappa^{\mathbb{P}*\dot{\mathbb{Q}}} \models \varphi$, then there are unboundedly many inaccessible cardinals κ with $V_\kappa \models \varphi$.

In this paper, I would like to present a more streamlined version of Chalons' axiom, along with an equiconsistency analysis of the principle in various forms. My intention with these streamlined axioms, which I call the Maximality Principles, is to take the essence of Chalons' idea to heart, and accord more closely with the plain language slogan, “*anything forceable and not subsequently unforceable is true.*”

So let me begin. Define that a sentence φ is *possible* or *forceable* when it holds in some forcing extension $V^\mathbb{P}$. The sentence φ is *necessary* when it holds in all forcing extensions $V^\mathbb{P}$ (by trivial forcing, this includes V). Thus, φ is *forceably necessary* (or *possibly necessary*) when φ holds in some forcing extension $V^\mathbb{P}$ and all subsequent extensions $V^{\mathbb{P}*\dot{\mathbb{Q}}}$, i.e. when it is forceable that φ is necessary.

The forceably necessary sentences are therefore those which can be turned ‘on’ in a permanent way, so that no further forcing can ever turn them off. Perhaps the simplest example of a forceably necessary sentence is $V \neq L$, since this is easy to force, and once it is forced, one cannot force its negation. More interesting examples include the assertion “*there is a real with minimal constructibility degree over L ,*” which can be forced via Sacks forcing—and once true, this assertion persists to any forcing extension—and the assertion “*there are a proper class of inaccessible cardinals, if any,*” which I consider in Theorem 8 below. Conversely, the sentences CH or \neg CH are forceable but never necessary because they can be forced true or false over any model of set theory. The principle I have in mind, or at least the first approximation to it, is the following:

Maximality Principle² (MP) Any statement in the language of set theory that is forceably necessary is true.

Here, I intend to assert MP as a scheme, the scheme that asserts of every statement φ in the language of set theory that

if φ is forceably necessary, then φ .

I mean, of course, that φ is true in V . But it actually follows from this scheme that φ must

²Chalons [3] considers another principle approaching this, a scheme that asserts that if a statement φ has the feature that for every inaccessible cardinal κ there is a poset $\mathbb{P} \in V_\kappa$ with $V_\kappa^\mathbb{P} \models \varphi$ and for every such \mathbb{P} and any further forcing $\dot{\mathbb{Q}} \in V_\kappa$ we have $V_\kappa^{\mathbb{P}*\dot{\mathbb{Q}}} \models \varphi$, then φ holds (in V). Using this scheme as an intermediary, Chalons establishes the consistency of the Chalons Maximality Principle by first establishing the consistency of it, assuming the existence of a measurable cardinal. The ideas of the proof of Theorem 5 in this paper show that one can get away with less; indeed, to obtain the consistency of Chalons' principle a Mahlo cardinal is more than sufficient, and actually it suffices to have $V_\delta \prec V$ for an inaccessible cardinal δ . Apart from this question, though, I prefer to untie the core idea of the expression *everything forceable and not unforceable is true* from the question of whether the statements are true in or forceable over every V_κ , and this seems to lead to the more direct principle I have stated here.

also be true in every forcing extension of V , that is, that φ is necessary; this is proved below in Theorem 1.

By Tarski's theorem on the non-definability of truth, there is in general no first order way to express whether a statement is forceable or true; and so there seems to be little hope of expressing the Maximality Principle scheme in a single first-order sentence of set theory. Thus, I remain content with the Maximality Principle expressed as a scheme.

As I hope my terminology suggests, the Maximality Principle admits a particularly natural expression in modal logic. Specifically, if we take $\Diamond\varphi$ to mean that φ is possible, that is, that φ is forceable, that it holds in some forcing extension, and $\Box\varphi$ to mean that φ is necessary, that is, that φ holds in all forcing extensions, then we have a very natural Kripke frame interpretation of modal logic in which the possible worlds are simply the models of ZFC and one world is accessible from another if it is a forcing extension of that world.³

In this Kripke frame, one can easily check many elementary modal facts. The assertion $\neg\Diamond\varphi \iff \Box\neg\varphi$, for example, expresses the trivial fact that φ is not forceable if and only if $\neg\varphi$ holds in all forcing extensions; furthermore, its validity provides a duality between \Diamond and \Box . The assertion $\Box(\varphi \implies \psi) \implies (\Box\varphi \implies \Box\psi)$ is similarly easy to verify in this context. Further, since every model of set theory is a (trivial) forcing extension of itself, we can conclude $\varphi \implies \Diamond\varphi$, asserting that any true statement is forceable (by trivial forcing), as well as the dual version $\Box\varphi \implies \varphi$, asserting that any necessary statement is true. Using the fact that a forcing extension of a forcing extension is a forcing extension of the ground model, we can verify $\Diamond\Diamond\varphi \implies \Diamond\varphi$, asserting that any statement forceable over a forcing extension is forceable over the ground model, and the dual version $\Box\varphi \implies \Box\Box\varphi$, asserting that if a statement holds in all forcing extensions, then this remains true in any forcing extension.

The axioms mentioned in the previous paragraph are exactly the axioms constituting the modal theory known as $S4$ (see, for example, [6]). And since the modal operators \Diamond and \Box , as I have defined them in this article, are expressible in the language of set theory, we can view these $S4$ axioms as simple theorems of ZFC.

In this modal terminology, the assertion that φ is forceably necessary is exactly expressed by the assertion $\Diamond\Box\varphi$. The Maximality Principle MP, therefore, asserts of every φ that

$$(\Diamond\Box\varphi) \implies \varphi.$$

This way of stating the axiom naturally leads one to the apparently stronger principle, known as the Euclidean Axiom in modal logic:

$$(\Diamond\Box\varphi) \implies \Box\varphi.$$

³A *Kripke frame* allows one to provide semantics for a modal theory. It consists of a collection of worlds and an accessibility relation between the worlds. In such a frame, an assertion is *possible* in a world if it is true in a world accessible by that world; the assertion is *necessary* in a world if it is true in all worlds accessible by that world. The accessibility relation of this article, the one relating any model of set theory with its forcing extensions, is both reflexive and transitive. Thus, it is what is known as an $S4$ frame.

As a scheme, however, this apparently stronger principle is in fact no stronger than the original principle, a fact I alluded to earlier. In fact, the Maximality Principle can be stated in a variety of equivalent forms:

Equivalent Forms Theorem 1 *The following schemes are equivalent, where φ ranges over all sentences in the language of set theory:*

1. *Every forceably necessary statement is true:* $\Diamond\Box\varphi \implies \varphi$.
2. *Every forceably necessary statement is necessary:* $\Diamond\Box\varphi \implies \Box\varphi$.
3. *Every forceable statement is necessarily forceable (i.e. forceable over every forcing extension):* $\Diamond\varphi \implies \Box\Diamond\varphi$.
4. *Every true statement is necessarily forceable:* $\varphi \implies \Box\Diamond\varphi$.
5. *Every non-necessary statement is necessarily non-necessary:* $\neg\Box\varphi \implies \Box\neg\Box\varphi$.

Proof: This is an elementary exercise in modal reasoning. Scheme 2 is actually a special case of scheme 1, obtained by replacing φ with $\Box\varphi$ in 1 to obtain $\Diamond\Box\Box\varphi \implies \Box\varphi$. This is just $\Diamond\Box\varphi \implies \Box\varphi$ because $\Box\Box\varphi$ is equivalent to $\Box\varphi$. Conversely, scheme 2 implies scheme 1 because $\Box\varphi$ implies φ . Scheme 3 is the contrapositive form of scheme 2 (with $\neg\varphi$ replacing φ) and vice versa, because $\neg\Diamond\Box\neg\varphi$ is equivalent to $\Box\Diamond\varphi$ and $\neg\Box\neg\varphi$ is equivalent to $\Diamond\varphi$. Scheme 4, similarly, is the contrapositive form of scheme 1. Finally, scheme 5 is obtained from scheme 3 (applied to $\neg\varphi$) by pushing the negation through to φ , and conversely by pulling it out again. \square

Schemes 2, 3 and 5 can be strengthened to the full equivalences

$$\Diamond\Box\varphi \iff \Box\varphi, \quad \Diamond\varphi \iff \Box\Diamond\varphi \quad \text{and} \quad \neg\Box\varphi \iff \Box\neg\Box\varphi$$

because the converse implication in each case is immediate.

Because the various schemes are equivalent, let me now freely refer to any of them as the Maximality Principle MP. In particular, in form 2 above the principle asserts that any possibly necessary statement is necessary, and thus, under the Maximality Principle the collection of statements holding necessarily is maximal.

Those readers familiar with modal logic will recognize Scheme 5 above, the axiom $\neg\Box \implies \Box\neg\Box\varphi$, as precisely the axiom one adds to $S4$ to make the modal theory $S5$ (see [6]). Therefore, since we have already observed that the weaker $S4$ axioms hold automatically, we can make the following conclusion:

Theorem 2 *In the Kripke frame whose worlds are the models of ZFC, each of which accesses precisely its forcing extensions, the Maximality Principle MP in a world is equivalent to the assertion of the $S5$ axioms of modal logic in that world.⁴*

⁴Let me emphasize here that the Maximality Principle includes the assertions $\Diamond\Box\varphi \implies \Box\varphi$ only for sentences φ . In particular, it would be wrong to include, as one might want in other contexts, the universal generalizations of this axiom applied to formulas with free variables. Indeed, doing so would result in an inconsistent theory, as I prove in Observation 3.

One naturally wants to express the Maximality Principle in a language with parameters, and so I will also consider the following boldface version of the Maximality Principle, which allows for arbitrary parameters from \mathbb{R} :

Maximality Principle, Boldface Version (\mathbb{MP}) *Any statement in the language of set theory with arbitrary real parameters that is forceably necessary is necessary. That is, $(\Diamond \Box \varphi) \implies \Box \varphi$.*

This boldface version of the principle also admits the equivalent forms of Theorem 1. Subsuming the parameters into a universal quantifier, one can alternatively think of the boldface Maximality Principle as the scheme asserting all sentences of the form

$$\forall x \in \mathbb{R} [\Diamond \Box \varphi(x) \implies \Box \varphi(x)],$$

where φ is a formula with one free variable x .

With real parameters, of course, one can in effect refer to any hereditarily countable set as a parameter. And while one might want to include even more parameters than this, it would be inadvisable to do so in view of the following:

Observation 3 *If a set p is not hereditarily countable, then there is a statement $\varphi(p)$ in the language of set theory with parameter p which is forceably necessary, but not true. Hence, the Maximality Principle asserted in the language of set theory with arbitrary parameters is false.*

Proof: Let $\varphi(p)$ simply be the statement, “ p is hereditarily countable.” Of course, this is forceably necessary, since any set can be made countable by forcing, and once countable, it can never be made uncountable again by further forcing. But by the hypothesis on p , the statement $\varphi(p)$ is not true. \square

Before introducing the next principle, let me observe the following.

Observation 4 *The Maximality Principle \mathbb{MP} , if true, is necessary. In modal terminology, $\mathbb{MP} \implies \Box \mathbb{MP}$.*

Proof: If a sentence φ is forceably necessary over a forcing extension, then it was already forceably necessary over the ground model, and if necessary in the ground model, it remains necessary in any forcing extension. That is, $\Diamond \varphi$ is downward absolute from a forcing extension, and $\Box \varphi$ is upward absolute to any forcing extension. So $\mathbb{MP} \implies \Box \mathbb{MP}$. \square

Observation 4 does not easily generalize to the boldface Maximality Principle \mathbb{MP} . The problem is that there can be new real parameters in the forcing extension and thus entirely new assertions φ there, which cannot be expressed in the ground model. This leads us to the strongest principle I will consider in this article, the principle that asserts:

Necessary Maximality Principle ($\Box \mathbb{MP}$) *The principle \mathbb{MP} is necessary; that is, it holds in V and every forcing extension $V^{\mathbb{P}}$.*

Let me caution the reader that one does not obtain the principle $\Box\mathcal{MP}$ merely by prefacing every assertion in the \mathcal{MP} scheme with a \Box , for this would only assert the scheme in a forcing extension using parameters from the ground model; such assertions hold for free by \mathcal{MP} as in Observation 4. Rather, $\Box\mathcal{MP}$ asserts that the full \mathcal{MP} scheme holds in every forcing extension $V^{\mathbb{P}}$, using the additional real parameters available there.

Later, I will prove that the principle $\Box\mathcal{MP}$ is much stronger by far than the other Maximality Principles. If it holds, for example, then $0^\#$ exists and more, the universe is closed under sharps and projective truth is invariant by set forcing. An upper bound on the consistency strength of $\Box\mathcal{MP}$ is not yet known; it may simply be false.

2 Consistency of the Maximality Principle \mathcal{MP}

Let us now analyze the consistency of the simplest of the Maximality Principles, namely, the lightface version \mathcal{MP} with no parameters.

Theorem 5 *If ZFC is consistent, then so also is ZFC plus the Maximality Principle \mathcal{MP} .*

Proof: In order to highlight various aspects of the theory, I will actually give two proofs of this theorem. The first proof is elementary, but the second proof will generalize to \mathcal{MP} . We begin with a simple observation.

Lemma 5.1 *For no sentence φ are both φ and $\neg\varphi$ forceably necessary.*

Proof: Suppose that φ is forced to be necessary by \mathbb{P} , so that φ holds in every forcing extension of $V^{\mathbb{P}}$, and that $\neg\varphi$ is forced necessary by \mathbb{Q} , so that $\neg\varphi$ holds in every forcing extension of $V^{\mathbb{Q}}$. Consider now the forcing $\mathbb{P} \times \mathbb{Q}$. Since this is isomorphic to $\mathbb{Q} \times \mathbb{P}$, the extension $V^{\mathbb{P} \times \mathbb{Q}}$ is a forcing extension both of $V^{\mathbb{P}}$ and of $V^{\mathbb{Q}}$. Thus, both φ and $\neg\varphi$ hold there, a contradiction. \square

This argument generalizes to the following.⁵

Lemma 5.2 *Over any model of set theory, the collection of forceably necessary statements forms a consistent theory.*

Proof: Consider any finite collection $\varphi_1, \varphi_2, \dots, \varphi_n$ of such sentences, each forceably necessary over a fixed model M . Suppose that φ_i is forced necessary by \mathbb{P}_i , and consider the partial order $P_1 \times \dots \times P_n$. The resulting forcing extension $M^{\mathbb{P}_1 \times \dots \times \mathbb{P}_n}$ is a forcing extension of each $M^{\mathbb{P}_i}$, and so each φ_i holds there. Thus, any given finite collection of sentences forceably necessary over a fixed model of set theory is consistent. And so the whole collection is consistent. \square

⁵Modal logicians will recognize that these lemmas rely on the *directedness* of the accessibility relation on worlds. Specifically, the crucial fact at play here is that any two forcing extensions can be combined into a third, which is a forcing extension of each of them.

Lemma 5.3 *If φ is forceably necessary, then this is necessary; and if φ is not forceably necessary, then this also is necessary. In short, the collection of forceably necessary sentences is invariant by forcing.*

Proof: Suppose that φ is forced necessary by the forcing \mathbb{P} , so that φ holds in every forcing extension of $V^{\mathbb{P}}$. Given any other forcing extension $V^{\mathbb{Q}}$, one can still force with \mathbb{P} to obtain $V^{\mathbb{Q} \times \mathbb{P}}$. Since this is the same as $V^{\mathbb{P} \times \mathbb{Q}}$, it is a forcing extension of $V^{\mathbb{P}}$. Thus, over $V^{\mathbb{Q}}$ one could still force φ to be necessary; and so φ is forceably necessary in $V^{\mathbb{Q}}$, and hence necessarily forceably necessary in V . Conversely, suppose that φ is not forceably necessary. This means that $\neg\varphi$ is forceable over every forcing extension. Thus, over no forcing extension can φ be forced necessary, so φ is necessarily not forceably necessary. In summary, whether a statement is forceably necessary or not cannot be affected by forcing. \square

To prove the theorem, now, suppose that M is any model of ZFC. Let T be the collection of sentences that are forceably necessary in M . This includes every axiom of ZFC, since these axioms hold in every forcing extension of M . By Lemma 5.2, the theory T is consistent, and so it has a model N . Suppose now that φ is forceably necessary over N ; I aim to show that φ is true in N . First, I claim that φ is actually in T . If not, then it is not forceably necessary in M , and so by Lemma 5.3 the sentence $\neg \Diamond \Box \varphi$ asserting that φ is not forceably necessary is necessary and hence in T . In this case, since N is a model of T , the sentence φ cannot be forceably necessary in N , contrary to our assumption. Thus, φ must be in T , and so it holds in N , as desired. \square

A Second Proof: I would like now to give an alternative proof of Theorem 5, relying on a more traditional iterated forcing construction. It is this iterated forcing argument that will generalize to the case of $\mathfrak{M}_{\mathbb{P}}$.

Let me motivate the first lemma by mentioning that if there is an inaccessible cardinal κ , then by the proof of the downward Lowenheim-Skolem Theorem, one can find a closed unbounded set $C \subseteq \kappa$ of cardinals δ with $V_\delta \prec V_\kappa$. The structure V_κ , therefore, is a model of ZFC with a cardinal δ such that V_δ is an elementary substructure of the universe. Axiomatizing this situation, let the expression “ $V_\delta \prec V$ ” represent the scheme, in the language with an additional constant symbol for δ , which asserts of any statement φ in the language of set theory that

$$\text{for every } x \in V_\delta, \text{ if } V_\delta \models \varphi[x], \text{ then } \varphi(x).$$

Each such assertion in this scheme is first order, since one need only refer to satisfaction in a set structure, V_δ , and this is provided by Tarski’s definition of the satisfaction relation. The little argument just given shows that if there is an inaccessible cardinal, then there is a model of ZFC satisfying the scheme $V_\delta \prec V$; indeed, since the club C provides whole towers of such δ , the consistency strength of $V_\delta \prec V$ is easily seen to be strictly less than the consistency of the existence of an inaccessible cardinal.

The really amazing thing, however, is that in fact $\text{ZFC} + V_\delta \prec V$ is equiconsistent with ZFC. One might incorrectly guess that if ZFC holds in V and the scheme $V_\delta \prec V$ holds, then

V knows that V_δ is a model of ZFC; but this conclusion would confuse the ‘external’ ZFC with the ‘internal’ ZFC of the model. What follows is only that V_δ satisfies any particular instance of an axiom of ZFC but not the formula asserting that V_δ satisfies the entire scheme ZFC. This subtle distinction is crucial, in view of the following elementary fact.⁶

Lemma 5.4 *If ZFC is consistent, then so is $\text{ZFC} + V_\delta \prec V$.*

Proof: Assume that ZFC is consistent; so it has a model M . By the Lévy Reflection Theorem, every finite subcollection of the theory $\text{ZFC} + V_\delta \prec V$ is modeled in some rank initial segment of M , and therefore is consistent. So the whole theory is consistent. \square

The theorem is now a consequence of the following lemma.

Lemma 5.5 *Assume that $V_\delta \prec V$. Then there is a forcing extension of V , by forcing of size at most δ , in which the Maximality Principle MP holds.*

Proof: Since the Maximality Principle asserts, in a sense, that all possible switches have been turned permanently on that are possible to turn permanently on, the idea behind the proof of this lemma will be an iteration that forces all statements that are possible to force in a permanent way. The trick is to handle the meta-mathematical issues that arise on account of the non-definability of what is true or forceable. It is in doing this that the hypothesis $V_\delta \prec V$ will be used.

Let $\langle \varphi_n \mid n \in \omega \rangle$ enumerate all sentences in the language of set theory. I will define a certain forcing notion \mathbb{P} , an ω -iteration, all of whose initial segments \mathbb{P}_n will be elements of V_δ . So, suppose recursively that the iteration \mathbb{P}_n has been defined up to stage n , and consider the model $V_\delta^{\mathbb{P}_n}$. If φ_n is forceably necessary over this model, then choose a poset \mathbb{Q}_n which forces it to be necessary, and let $\mathbb{P}_{n+1} = \mathbb{P}_n * \mathbb{Q}_n$ using a suitable name for \mathbb{Q}_n . Let \mathbb{P} be the finite support iteration of the \mathbb{P}_n .

I claim that if $G \subseteq \mathbb{P}$ is V -generic, then the Maximality Principle holds in $V[G]$. To see this, suppose that φ is forceably necessary over $V[G]$. Necessarily, φ is φ_n for some n . Factor the forcing \mathbb{P} at stage n as $\mathbb{P} \cong \mathbb{P}_n * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is the iteration after stage n . Since φ is forceably necessary over $V[G] = V[G_n][G_{\text{tail}}]$, it is also forceably necessary over $V[G_n]$, because $V[G]$ is a forcing extension of $V[G_n]$ and so any forcing extension of $V[G]$ is also a forcing extension of $V[G_n]$. Since $V_\delta \prec V$ and $\mathbb{P}_n \in V_\delta$, whether a condition in \mathbb{P}_n forces a given statement has the same answer in V as in V_δ . Consequently, $V_\delta[G_n] \prec V[G_n]$. Thus, φ_n is forceably necessary over $V_\delta[G_n]$. In this case, the stage n forcing \mathbb{Q}_n forced it to be necessary, so φ holds in $V_\delta[G_{n+1}]$ and all its forcing extensions. By elementarity again, φ holds in $V[G_{n+1}]$ and all forcing extensions. In particular, since $V[G]$ is a forcing extension of $V[G_{n+1}]$, it holds in $V[G]$ and all further extensions. That is, φ is necessary in $V[G]$. \square

This completes the second proof of Theorem 5. \square

⁶Thanks to Ali Enayat for a helpful discussion of these ideas.

Corollary 6 *The theory $\text{ZFC} + \text{MP}$ is equiconsistent with ZFC .*

Though the hypothesis $V_\delta \prec V$ has now dropped away, there is nevertheless a stronger connection between $\text{ZFC} + \text{MP}$ and $\text{ZFC} + V_\delta \prec V$ than the previous corollary might suggest, a connection I would like to explore in the next few theorems. One cannot, for example, omit the hypothesis that $V_\delta \prec V$ from Lemma 5.5.

Theorem 7 *If ZFC is consistent, then there is a model of ZFC having no forcing extension, and indeed, no extension of any kind with the same ordinals, that is a model of MP . Thus, one cannot omit the hypothesis that $V_\delta \prec V$ from Lemma 5.5.*

Proof: I claim, first, that if there is a model of ZFC , then there is one in which the definable ordinals—and I mean ordinals that are definable without parameters—are unbounded. To see this, suppose that W is a model of ZFC , not necessarily transitive, and let M be the cut determined by the definable elements of W . That is, M consists of those elements in $(V_\alpha)^W$, where α ranges over the definable ordinals of W . I will show that $M \prec W$, by verifying the Tarski-Vaught criterion: suppose $W \models \exists x \varphi(x, a)$ for some $a \in M$. By the definition of M , it must be that a is in some V_α in the sense of W for some definable ordinal α of W . In W , let β be the least ordinal such that for every $y \in V_\alpha$ there is an $x \in V_\beta$ such that $\varphi(x, y)$, if there is any such x in W at all. Such a β exists by the Replacement Axiom. Moreover, β is definable in W , and consequently, $(V_\beta)^W$ is included in M , and so M has all the witnesses it needs to verify the Tarski-Vaught criterion. Thus, $M \prec W$, and in particular, M is a model of ZFC whose definable elements are unbounded. By beginning with a model W that also satisfies $V = L$, we may assume additionally that M satisfies $V = L$ as well.

Suppose now, towards contradiction, that M has an extension N , with the same ordinals as M , that satisfies $\text{ZFC} + \text{MP}$. Necessarily, $M = (L)^N$. Consider any definable ordinal α of W , defined by the formula ψ . Since $M \prec W$, the formula ψ also defines α in M , and since M is the L of N , we know further that α is definable in N as “the ordinal satisfying ψ in L ”. Let φ be the sentence expressing, “the ordinal defined by ψ in L is countable”. This sentence is forceably necessary over N , since if α is not countable in N , one can force to make it countable, and having done this, of course, it remains countable in any further extension. Thus, by MP , the sentence φ must be already true in N , and so α is countable in N .

The key observation is now that since the definable ordinals α of M were unbounded in M , it must be that every ordinal of M is countable in N , a contradiction, since these two models of ZFC have the same ordinals. \square

While Lemma 5.4 establishes the equiconsistency of ZFC with $\text{ZFC} + V_\delta \prec V$, when it comes to transitive models the two theories have different strengths. Specifically, if M is an ω -model of $V_\delta \prec V$, then M has the correct internal ZFC , and so since every axiom of ZFC is true in V_δ , the model M agrees that V_δ is a model of ZFC . Consequently, inside any \in -minimal transitive model of $\text{ZFC} + V_\delta \prec V$ there is a transitive model of ZFC but, by minimality, no transitive model of $\text{ZFC} + V_\delta \prec V$. So it is relatively consistent that there

is a transitive model of the one theory, but not the other. In this sense, we might say that the theory $\text{ZFC} + V_\delta \prec V$ has a greater transitive-model consistency strength than ZFC . The next theorem shows that the transitive-model consistency strength of $\text{ZFC} + \text{MP}$ is similarly greater than ZFC , since in fact it is the same as $\text{ZFC} + V_\delta \prec V$.

Theorem 8 *The theories $\text{ZFC} + \text{MP}$ and $\text{ZFC} + V_\delta \prec V$ are transitive-model-equiconsistent in the sense that one of them has a transitive model if and only if the other also does. Indeed, for any transitive model of one of these theories, there is a model of the other with the same ordinals.*

Proof: Lemma 5.5 shows that every model of $\text{ZFC} + V_\delta \prec V$ has a forcing extension that is a model of $\text{ZFC} + \text{MP}$.

Conversely, suppose that M is a transitive model of $\text{ZFC} + \text{MP}$. Consider the L of M , which must have the form L_γ , for $\gamma = \text{ORD}^M$. The arguments of the previous theorem establish that every ordinal that is definable in L^M is countable in M . Let δ be the supremum of such ordinals. Thus, δ is at most ω_1^M , and in particular, $\delta < \gamma$. The initial claim of the previous theorem establishes that in the model L^M , the cut determined by the definable ordinals is an elementary substructure. In this case, this means that $V_\delta \prec V$ in L^M , as desired. \square

The argument establishes that for an arbitrary order-type ξ (not necessarily well-ordered), the ξ -consistency strength of $\text{ZFC} + V_\delta \prec V$ is at least as great as that of $\text{ZFC} + \text{MP}$, since from every model of the former theory we can build a model of the latter theory with the same ordinals. The argument, however, does not quite establish the converse, because in the non-transitive case one doesn't seem to know that the supremum of the definable ordinals of L^M exists as an ordinal of M .

In the proof of Lemma 5.5, the forcing \mathbb{P} may have size δ , and in this case it may happen that every cardinal below δ may be collapsed to ω , so there is little reason to expect that the axiom $V_\delta \prec V$ remains true in the forcing extension $V[G]$ (that is, using $(V[G])_\delta$ and $V[G]$). By taking a little care, however, the following theorem explains how to arrange $\text{ZFC} + \text{MP} + V_\delta \prec V$ in the extension.

Theorem 9 *If ZFC is consistent, then so is the theory $\text{ZFC} + \text{MP} + V_\delta \prec V$.*

Proof: First, one shows that if there is a model of ZFC , then there is a model of $\text{ZFC} + V_\delta \prec V + \text{cof}(\delta)$ is uncountable. This can be proved in the same way as Lemma 5.4, by simply adding the assertion “ $\text{cof}(\delta) > \omega$ ” to the theory. The Lévy Reflection Theorem shows that any finite collection of formulas is absolute from V to V_δ for a closed unbounded class of cardinals, and so one may find such a δ of uncountable cofinality.

Given a model of $\text{ZFC} + V_\delta \prec V$ in which $\text{cof}(\delta)$ is uncountable, one now proceeds with the argument of Lemma 5.5, using finite support in the iteration \mathbb{P} of that theorem. Since on cofinality grounds \mathbb{P} has size less than δ , it follows that $V_\delta[G] \prec V[G]$ in that argument. That is, the axiom “ $V_\delta \prec V$ ” remains true in $V[G]$, together with $\text{ZFC} + \text{MP}$. \square

3 Consistency of the Maximality Principle $\mathbb{M}\mathbb{P}$

Let me now turn to the boldface Maximality Principle $\mathbb{M}\mathbb{P}$, in which arbitrary real parameters are allowed to appear in the formulas φ of the scheme $\Diamond \Box \varphi \implies \Box \varphi$. As one might expect, the boldface principle $\mathbb{M}\mathbb{P}$ turns out to have a stronger consistency strength.

Define the expression “ORD is Mahlo” to be the scheme asserting that any closed unbounded class $C \subseteq \text{ORD}$ that is definable from parameters contains a regular cardinal. This is also sometimes referred to as the Lévy Scheme (e.g., see [5]). Whenever κ is a Mahlo cardinal, then $V_\kappa \models \text{ZFC} + \text{ORD is Mahlo}$; so the consistency strength of “ORD is Mahlo” is strictly less than the existence of a Mahlo cardinal. The converse of this implication, however, is not generally true, because if κ is a Mahlo cardinal, then there is a club $C \subseteq \kappa$ consisting of δ with $V_\delta \prec V$. These V_δ will therefore also model “ZFC + ORD is Mahlo”, and so there will be many non-Mahlo cardinals with that property.

Theorem 10 *The following theories are equiconsistent:*

1. ZFC plus the Maximality Principle $\mathbb{M}\mathbb{P}$.
2. ZFC plus $V_\delta \prec V$ for an inaccessible cardinal δ .
3. ZFC plus ORD is Mahlo.

Proof: I will prove the theorem with a sequence of lemmas.

Lemma 10.1 *Every model of theory 1 has an inner model of theory 2. Specifically, if $\mathbb{M}\mathbb{P}$ holds, then $\delta = \omega_1$ is inaccessible in L , and $L_\delta \prec L$.*

Proof: Assume $\mathbb{M}\mathbb{P}$ and let $\delta = \omega_1$. I claim, first, that δ is inaccessible to reals. To see this, suppose x is a real, and let φ be the statement “the ω_1 of $L[x]$ is countable”. Since this is forceably necessary, by $\mathbb{M}\mathbb{P}$ it is true. Thus, $\omega_1^{L[x]} < \delta$, as I claimed. It follows from this that δ is inaccessible in L .

Second, I claim that L_δ is an elementary substructure of L (note: I prove this part of the Lemma only as a scheme). To see this, simply verify the Tarski-Vaught criterion: suppose that L satisfies $\exists y \psi(a, y)$ for a set a in L_δ . Let α be least such that there is such a y in L_α . Consider the statement φ asserting “the least α , such that there is a y in L_α with $\psi(a, y)^L$, is countable”. This is expressed using the parameter a , which is coded with a real. Since it is forceably necessary, it must be true by $\mathbb{M}\mathbb{P}$, and so α is countable. Thus, y can be found in L_δ , and so the Tarski-Vaught criterion holds. In summary, L satisfies theory 2. \square

Lemma 10.2 *If theory 2 holds, then theory 1 holds in a forcing extension. Indeed, if $V_\delta \prec V$ and δ is an inaccessible cardinal, then after the Lévy collapse making δ the ω_1 of the extension, the Maximality Principle $\mathbb{M}\mathbb{P}$ holds.*

Proof: Assume that $V_\delta \prec V$ and δ is inaccessible. In order to obtain $\mathbb{M}\mathbb{P}$ in a forcing extension, I will define a certain finite support δ -iteration \mathbb{P} which at every stage forces with some poset of rank less than δ . Let $\langle \varphi_\alpha \mid \alpha < \delta \rangle$ enumerate, with unbounded repetition, all possible sentences in the forcing language for initial segments of such iterations with parameters having names in V_δ . Suppose now that the iteration \mathbb{P}_α is defined up to stage α ; I would like to define the stage α forcing $\dot{\mathbb{Q}}_\alpha$. If φ_α is a sentence with parameters naming objects in $V_\delta^{\mathbb{P}_\alpha}$, and if it is possible to force over $V_\delta^{\mathbb{P}_\alpha}$ so as to make φ_α necessary, then choose $\dot{\mathbb{Q}}_\alpha \in V_\delta$ to be (the name of) a poset accomplishing this. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be trivial forcing. This defines the δ -iteration \mathbb{P} .

Suppose that $G \subseteq \mathbb{P}$ is V -generic, and consider the model $V[G]$. First observe that the cardinal δ is not collapsed by this forcing, since it is a finite support iteration of δ -c.c. forcing. Consequently, every real in $V[G]$ appears in some $V[G_\alpha]$. Thus, if φ is a sentence with (names for) real parameters from $V[G]$, it must be φ_α for some α , with the names in φ coming from $V[G_\alpha]$. So it suffices to consider the formulas φ_α .

Suppose that such a φ_α is forceably necessary over $V[G]$. Then it is also forceably necessary over $V[G_\alpha]$, since $V[G]$ is a forcing extension of $V[G_\alpha]$. Hence, by the elementarity $V_\delta[G_\alpha] \prec V[G_\alpha]$ (as in Lemma 5.5), the sentence φ_α is forceably necessary over $V_\delta[G_\alpha]$. The stage α forcing $\dot{\mathbb{Q}}_\alpha$ must then have forced it to be necessary, and so φ_α became necessary in $V_\delta[G_{\alpha+1}]$, and hence also in $V[G_{\alpha+1}]$. Thus, since $V[G]$ is obtained by further forcing over this model, φ_α remains necessary in $V[G]$. So $V[G]$ satisfies $\mathbb{M}\mathbb{P}$.

Finally, let me consider the precise nature of the iteration \mathbb{P} ; in fact, we know \mathbb{P} very well. Since the assertion “ a is countable” is always forceably necessary, it follows that every element $a \in V_\delta$ is countable in $V[G]$. That is, unboundedly often, the forcing $\dot{\mathbb{Q}}_\alpha$ collapses elements of V_δ . Since δ itself is not collapsed, it becomes the ω_1 of $V[G]$. But up to isomorphism, the only finite support δ -iteration of forcing notions of size less than δ that collapses all cardinals below δ to ω is the Lévy collapse of δ to ω_1 . Thus, \mathbb{P} is isomorphic to the Lévy collapse, as I claimed. \square

David Asperó has used a similar iteration to prove a similar conclusion relating to his generalized bounded forcing axioms (see Theorem 3.7 in [1]).

Lemma 10.3 *Theory 2 implies theory 3.*

Proof: Suppose that $V_\delta \prec V$ and δ is inaccessible. If $C \subseteq \text{ORD}$ is a definable club, then of course $C \cap \delta$ is unbounded in δ . Thus, $\delta \in C$, and so C contains a regular cardinal, as desired. \square

Lemma 10.4 *If theory 3 is consistent, then so is theory 2.*

Proof: This argument is similar to Lemma 5.4. Suppose that M is a model of ZFC plus the scheme asserting that ORD is Mahlo. By the Lévy Reflection Theorem, any finite collection of formulas reflects from V to V_θ for a closed unbounded class of cardinals θ . Thus, since ORD

is Mahlo in M , we can find in M a regular cardinal δ such that the formulas are absolute between V and V_δ in M . We may assume that the finite collection of formulas includes the assertion that for every β , the cardinal 2^β exists, and from this it follows that δ is a strong limit cardinal. Thus, δ is actually inaccessible in M . So every finite subcollection of theory 2 is consistent, and so the entire theory is consistent. \square

This completes the proof of the theorem. \square

Lemmas 10.1 and 10.2 establish that if there is a model of one of the two theories “ZFC + \mathfrak{MP} ” or “ZFC + $V_\delta \prec V + \delta$ is inaccessible,” then there is a model of the other with the same ordinals. Therefore, we may conclude:

Corollary 11 *The theories “ZFC + \mathfrak{MP} ” and “ZFC + $V_\delta \prec V + \delta$ is inaccessible” are ξ -equiconsistent for every order type ξ .*

Since Lemma 10.2 proceeds via the Lévy collapse, one naturally obtains the CH in the extension, and this leads one to wonder whether $\mathfrak{MP} + \neg\text{CH}$ is consistent. The corresponding question does not arise with the lightface Maximality Principle MP because once MP is true, it persists to all forcing extensions, and so one can simply force CH or $\neg\text{CH}$ as desired, while preserving MP. But with the boldface Maximality Principle, it is not clear that \mathfrak{MP} persists to an extension where new real parameters are available. The following theorem shows how to sidestep this worry.

Theorem 12 *The boldface Maximality Principle \mathfrak{MP} is relatively consistent with either CH or $\neg\text{CH}$.*

Proof: Over any model one can force CH without adding reals, and this will preserve \mathfrak{MP} . Alternatively, the model of Lemma 10.2 satisfies $\mathfrak{MP} + \text{CH}$, since it is obtained via the Lévy collapse of an inaccessible cardinal.

To obtain a model of $\mathfrak{MP} + \neg\text{CH}$, we will simply add Cohen reals over this model.⁷ Specifically, I claim that the model $V[G][g]$, where $G \subseteq \mathbb{P}$ is V -generic for the iteration of Lemma 10.2 and g is $V[G]$ -generic for the forcing to add at least $\omega_2^{V[G]}$ many Cohen reals, is a model of $\mathfrak{MP} + \neg\text{CH}$. Certainly $\neg\text{CH}$ is no problem, so consider \mathfrak{MP} . The instances of the axiom scheme $\diamond \Box \varphi \implies \Box \varphi$, where φ has real parameters from $V[G]$, hold in $V[G]$ by Lemma 10.2, and each of these instances persists to every forcing extension, including $V[G][g]$. At issue is whether the scheme holds of formulas using the new parameters available in $V[G][g]$. Consider, therefore, a real x in $V[G][g]$ and the corresponding extension $V[G][x]$. Since x is added by a countable forcing notion \mathbb{Q} , we may reorganize this forcing as $V[G][x] = V[x][G']$, where x is now V -generic for some countable forcing notion and G' is $V[x]$ -generic for the Lévy collapse of κ in $V[x]$ (this is because the quotient forcing $(\mathbb{P} * \mathbb{Q})/x$ has size κ , is κ -c.c. and collapses every cardinal below κ to ω ; hence it is the Lévy collapse). Since $V_\kappa[x] \prec V[x]$, it follows by Lemma 10.2 that \mathfrak{MP} holds in $V[x][G']$. Thus, instances of the \mathfrak{MP} scheme using

⁷I am grateful to Ilijas Farah for pointing out the applicability of Solovay’s technique here.

the parameter x hold in $V[x][G'] = V[G][x]$. Since these persist to any forcing extension, they also hold in $V[G][g]$. So \mathbb{MP} holds in $V[G][g]$, as desired. \square

Techniques of Kunen, employing the same reorganization idea as above, can be used to obtain a model of $\mathbb{MP} + \neg\text{CH} + \text{MA}$.

Let me close this section with a discussion of the relative consistency of \mathbb{MP} with large cardinals.

Theorem 13 *The Maximality Principles \mathbb{MP} and \mathbb{MP} are consistent with the existence of inaccessible cardinals, measurable cardinals, supercompact cardinals, or what have you, assuming the consistency of a proper class of such cardinals initially (plus, for \mathbb{MP} , that ORD is Mahlo). On the other hand, the Maximality Principle is also consistent with the nonexistence of such cardinals, assuming only the consistency of ZFC.*

Proof: Suppose that ZFC is consistent with the existence of a proper class of, for example, measurable cardinals. The argument of Lemma 5.4 produces a model of $\text{ZFC} + V_\delta \prec V$ with a proper class of measurable cardinals. Since the forcing of Lemma 5.5 has size at most δ , the measurable cardinals above δ survive to the forcing extension in which \mathbb{MP} holds. For \mathbb{MP} , one should add the hypothesis that “ORD is Mahlo” to this argument in order to get a model as above for which δ is inaccessible, and then carry on as in Theorem 10. This procedure shows that any large cardinal that is not destroyed by small forcing will be consistent with \mathbb{MP} or \mathbb{MP} .

To obtain the Maximality Principle in the absence of all large cardinals, one should simply apply the argument of Theorem 5 using a model with no inaccessible cardinals. That is, if M is any model of ZFC with no inaccessible cardinals, then no forcing extension of M will have inaccessible cardinals, and so the assertion that there are no inaccessible cardinals will be necessary in M and hence in the theory T of that proof. \square

The attentive reader will observe that in order to deduce the consistency of the Maximality Principle with an inaccessible cardinal, the previous proof begins with and eventually produces models with proper classes of such cardinals. This apparent inefficiency suggests the question of whether the Maximality Principles are consistent with the existence of a single inaccessible cardinal. The answer, perhaps surprising at first, is that they are not.

Theorem 14 *If the Maximality Principle \mathbb{MP} holds, then there are a proper class of inaccessible cardinals, if any.*⁸

Proof: Suppose that the Maximality Principle holds, but that the inaccessible cardinals are bounded. Then the assertion “there are no inaccessible cardinals” is forceably necessary, because one could force, in a permanent way, to make all the inaccessible cardinals countable. Thus, under \mathbb{MP} , there must have been no inaccessible cardinals to begin with. \square

⁸Chalons [3] uses a similar idea when proving that the Chalons Maximality Principle does not follow from any large cardinal axiom.

The same argument works with any other large cardinal whose existence is downward absolute from a forcing extension, such as Mahlo cardinals. For measurable cardinals, whose existence is not downward absolute in this way, the situation is slightly more complicated. But one can define that a cardinal κ is *potentially* measurable when it is measurable in a forcing extension. This notion, of course, is downward absolute from any forcing extension. Consequently, if MP holds and there is a potentially measurable cardinal, then there must be a proper class of potentially measurable cardinals. (If there are a bounded number, then force to collapse them all, making none, in a permanent way; so there must have been none to begin with.) An identical fact holds for potentially supercompact cardinals, or what have you.

4 The Necessary Maximality Principle

We turn now to the strongest principle I have mentioned in this paper, the Necessary Maximality Principle $\square_{\mathcal{M}\mathcal{P}}$. I will begin with a theorem hinting at surprising strength.

Theorem 15 *If $\square_{\mathcal{M}\mathcal{P}}$ holds, then $0^\#$ exists.*

Proof: It suffices, by the Covering Lemma of Jensen, to find a violation of covering. For this, it suffices to show that L fails to compute the successors of singular cardinals correctly. In fact, I will show that L fails to compute the successor of any cardinal correctly. Consider any cardinal κ and its successors $(\kappa^+)^L$ and $(\kappa^+)^V$ in the two models. Let g be V -generic for the canonical forcing that collapses κ to ω . This forcing is the same in V or L and has size κ in either model, and so by the chain condition it preserves κ^+ over either model, making it the ω_1 of the extension. The key observation now is that the sentence φ , asserting “ $(\omega_1)^{L[g]}$ is countable,” is forceably necessary. Thus, by $\mathcal{M}\mathcal{P}$ in $V[g]$, this sentence is true in $V[g]$. Putting these conclusions together, we have

$$(\kappa^+)^L = (\omega_1)^{L[g]} < (\omega_1)^{V[g]} = (\kappa^+)^V,$$

and so L does not compute the successor of κ correctly. \square

This argument generalizes to the following, where by W_γ I mean $(V_\gamma)^W$.

Theorem 16 *Suppose that $\square_{\mathcal{M}\mathcal{P}}$ holds and W is a definable transitive class, invariant under set forcing. Then:*

1. *The class W does not compute successor cardinals correctly.*
2. *Indeed, every regular cardinal is inaccessible in W .*
3. *Indeed, every cardinal is a limit of inaccessible cardinals in W .*
4. *For every cardinal γ , $W_\gamma \prec W$.*

5. For every cardinal γ , $H(\gamma)^W \prec W$.
6. For every cardinal γ , if a set x is definable in W from an object $a \in H(\gamma)^V$, then $x \in H(\gamma)^V$.

Proof: Clearly, statement 1 of the theorem is implied by statement 2, which is in turn implied by statement 3. Let me argue that statement 3 is implied by statement 4. If $W_\gamma \prec W$, then clearly γ is a strong limit in W , and hence, if regular, γ is inaccessible in W . Thus, the inaccessible cardinals are unbounded in W , and hence also in W_γ , so statement 3 holds. Statement 4 easily follows from statement 5, because if $H(\gamma)^W \prec W$, then $H(\gamma)^W$ has the correct $(V_\alpha)^W$ for $\alpha < \gamma$, and so $H(\gamma)^W = W_\gamma$, giving statement 4.

Next, let me show that statement 5 follows from statement 6, by verifying the Tarski-Vaught criterion. Suppose that W satisfies $\exists x \psi(x, a)$ where $a \in H(\gamma)^W$. The least hereditary size in W of such an x is definable in W from a , and so by statement 6 this size must be less than γ . In particular, there will be such an x in $H(\gamma)^W$, and so the criterion is fulfilled.

It remains only to prove statement 6 of the theorem. Suppose that x is definable in W from the object $a \in H(\gamma)$. Let $\beta < \gamma$ be the hereditary size of a , so that $a \in H(\beta^+)$, and suppose that $g \subseteq \beta$ is V -generic for the canonical forcing that collapses β to ω . Let φ be the sentence asserting “ x is hereditarily countable”, using the definition of x in W from a , coded with the real g . Thus, φ is a sentence with real parameters, and it is forceably necessary (please note that I am using here the invariance of W under set forcing in order to know that x is still defined in the same way in any forcing extension). Consequently, since $\square_{\mathbb{M}\mathbb{P}}$ implies $\mathbb{M}\mathbb{P}$ in $V[g]$, the sentence φ must be true in $V[g]$. That is, x is hereditarily countable in $V[g]$. Since β^+ and γ are preserved to $V[g]$, it follows that x has hereditary size less than γ in V , as desired. So the proof is complete. \square

The theorem is easily modified to handle the situation when W is defined from parameters. Indeed, if W is definable from a real parameter, the previous proof goes through essentially unchanged. More generally, we have:

Corollary 17 *Suppose that $\square_{\mathbb{M}\mathbb{P}}$ holds and W is a transitive class, definable from parameters $\vec{p} \in H(\eta^+)$, which is invariant under set forcing. Then the conclusions of the previous theorem hold above η .*

Proof: By this, I mean that such a class W will not compute successor cardinals above η correctly, that every cardinal above η will be a limit of inaccessible cardinals in W , and so on, and that every set x definable in W from an object in $H(\gamma)$ for some γ above η will be in $H(\gamma)$.

Suppose, therefore, that W is definable from the parameters $\vec{p} \in H(\eta^+)$, and that it is invariant by set forcing. In the forcing extension $V[g]$ that collapses η to ω , the parameters \vec{p} become hereditarily countable, and the class W therefore becomes definable from a real z . Thus, applying the previous proof to the class W , defined by the real z in $V[g]$, the

conclusion follows for the cardinals of $V[g]$. Since these are precisely the cardinals above η in V , the corollary is proved. \square

Corollary 18 *If $\square_{\mathcal{MP}}$ holds, then for every set x , the set $x^\#$ exists. That is, the universe is closed under sharps.*

Proof: Fix any set x , and let g be a real collapsing cardinals so that x can be coded by a real \bar{x} in $V[g]$. Let $W = L[x]$. This is definable from \bar{x} , and is invariant in any further extension of $V[g]$. Thus, by the previous corollary, if $\square_{\mathcal{MP}}$ holds, then $L[x]$ does not compute the successors of singular cardinals above x correctly. It follows that $x^\#$ exists in $V[g]$, and so it also exists in V . \square

The same idea shows that the universe is closed under daggers and pistols. Any model of the form $L[x, \mu]$, for example, where the measure μ lives on the smallest possible ordinal, is definable from x , and Theorem 16 shows that this model cannot compute successor cardinals correctly above x .

Projective Absoluteness Theorem 19 *The Necessary Maximality Principle $\square_{\mathcal{MP}}$ implies Projective Absoluteness. That is, if $\square_{\mathcal{MP}}$ holds, then projective truth is invariant by set forcing; every Σ_n^1 formula is absolute to any set forcing extension.*

Proof: Assume $\square_{\mathcal{MP}}$. I will show, by induction on n , that every Σ_n^1 formula φ is absolute to any set forcing extension. This is trivial for $n = 0$ (and indeed, it holds automatically up to Σ_2^1 by the Shoenfield Absoluteness Theorem). Assume by induction that it is true for all Σ_n^1 formulas. Further, since $\square_{\mathcal{MP}}$ holds in all set forcing extensions, suppose that this induction hypothesis also holds in all set forcing extensions, that is, that the Σ_n^1 formulas are absolute from any set forcing extension to any further set forcing extension. Since the collection of φ for which this hypothesis is true is clearly preserved under Boolean combinations, it suffices to consider an existential formula $\psi(x) = \exists y \varphi(x, y)$, where φ is absolute from any forcing extension to any further forcing extension. If $\psi(a)$ is true in such a model, then since φ is absolute to any set forcing extension and both a and the witness y still exist there, $\psi(a)$ remains true in any set forcing extension. Conversely, suppose that $\psi(a)$ is true in a set forcing extension $V^{\mathbb{P}}$, with $a \in V$. Thus, the witness y to φ was added by the forcing \mathbb{P} . Since by the induction hypothesis the formula $\varphi(a, y)$ is absolute to any further forcing extension, the formula $\psi(a)$ is necessary in $V^{\mathbb{P}}$ and therefore forceably necessary in V . Thus, by $\square_{\mathcal{MP}}$, it is already true in V , as desired. So every projective formula is absolute by set forcing, and the theorem is proved. \square

Corollary 20 *The consistency of $\square_{\mathcal{MP}}$ implies that of the existence of infinitely many strong cardinals.*

Proof: Kai Hauser [4] has shown that projective absoluteness is equiconsistent with the existence of infinitely many strong cardinals. \square

The next theorem hints at an even stronger strength for $\square_{\mathcal{MP}}$, assuming that ORD is ineffable, i.e., the scheme asserting that for every class partition $F : [\text{ORD}]^2 \rightarrow 2$ there is a homogeneous stationary class $H \subseteq \text{ORD}$. This axiom is naturally expressed in Gödel-Bernays set theory, using class quantifiers. I am indebted to Philip Welch for his helpful remarks concerning the next few corollaries.

Corollary 21 *If $\square_{\mathcal{MP}}$ holds and ORD is ineffable, then there is an inner model of a Woodin cardinal.*

Proof: Steel [7] has proved that if there is no inner model of a Woodin cardinal and if the universe is closed under sharps and ORD is ineffable, then there is a definable class K , now widely known as the Mitchell-Steel core model, which is invariant under set forcing and which computes the successors of singular cardinals correctly. Since Theorem 16 shows that if $\square_{\mathcal{MP}}$ holds, then the universe is closed under sharps but there can be no such class K , it follows under the hypothesis that there must be an inner model with a Woodin cardinal. \square

One can modify the assumption that ORD is ineffable by building the core model up to a weakly compact cardinal κ . Specifically, Steel [7] shows that if κ is weakly compact, V_κ is closed under sharps and V_κ has no inner model of a Woodin cardinal, then the core model K built up to κ will compute the successors of singular cardinals correctly. Thus, since \mathcal{MP} implies closure under sharps and refutes the possibility of such a class K , this establishes:

Corollary 22 *If κ is weakly compact and $V_\kappa \models \square_{\mathcal{MP}}$, then V_κ has an inner model of a Woodin cardinal.*

This kind of analysis can be lifted to those core models with more Woodin cardinals, provided that Weak Covering holds. Following this line, Philip Welch has announced the following:

Theorem 23 (Welch) *If $\square_{\mathcal{MP}}$, then Projective Determinacy PD holds.*

Welch also conjectured that if $\square_{\mathcal{MP}}$ holds and there is an inaccessible cardinal, then $\text{AD}^{L(\mathbb{R})}$. Woodin has announced, in conversation, that $\square_{\mathcal{MP}}$ implies $\text{AD}^{L(\mathbb{R})}$ outright.

I regret that the obvious problem remains unsolved:

Challenge Problem 24 *Is $\text{ZFC} + \square_{\mathcal{MP}}$ (relatively) consistent?*

I would welcome a consistency proof relative to any large cardinal hypothesis.

Let me present a theorem, suggested by W. Hugh Woodin, that makes at least some progress in this direction. Let's say that a sentence is *local* if it can be expressed in the form “ ψ holds in V_κ , where κ is the least inaccessible cardinal”. Such sentences assert a truth that can be checked locally, in this V_κ , without need to consult the entirety of V . This is a broad

class of assertions, including all the conjectures and theorems of classical mathematics, as well as the bulk of contemporary mathematics.

Please note that the least inaccessible cardinal, of course, is not invariant by forcing; a model may disagree with a forcing extension on the particular value of κ even when the two models agree about some particular local assertion.

Theorem 25 *If there are a proper class of Woodin cardinals, then the Necessary Maximality Principle $\square_{\mathcal{M}\mathcal{P}}$ holds for local assertions (using real parameters in any forcing extension).*

Proof: I will first prove that if there are a proper class of Woodin cardinals, then the Maximality Principle $\mathcal{M}\mathcal{P}$ holds for local assertions using real parameters in the ground model. From this, since the hypothesis of a proper class of Woodin cardinals is preserved by set forcing, it follows that the same conclusion holds in every forcing extension as well. That is, from the hypothesis of a proper class of Woodin cardinals one can conclude the Necessary Maximality Principle $\square_{\mathcal{M}\mathcal{P}}$ for local assertions as well.

To begin, then, suppose that φ is local and forceably necessary, forced necessary by the forcing \mathbb{P} , with real parameters in V . Choose a Woodin cardinal δ above the rank of \mathbb{P} and let $\mathbb{P}_{<\delta}$ be the stationary tower forcing corresponding to δ . Suppose that $G_{<\delta} \subseteq \mathbb{P}_{<\delta}$ is V -generic below a condition forcing that $2^{|\mathbb{P}|}$ becomes countable. By the basic properties of stationary tower forcing (see, e.g. [8], [9]), there is a generic elementary embedding $j : V \rightarrow M \subseteq V[G_{<\delta}]$, with $M^{<\delta} \subseteq M$ in $V[G_{<\delta}]$. The model M therefore agrees with $V[G_{<\delta}]$ well beyond the first inaccessible cardinal. Since furthermore $2^{|\mathbb{P}|}$ was collapsed, in $V[G_{<\delta}]$ we may construct a V -generic filter for \mathbb{P} , and therefore we may view $V[G_{<\delta}]$ as a forcing extension of $V^{\mathbb{P}}$. Since φ is made necessary in $V^{\mathbb{P}}$, it follows that φ holds in $V[G_{<\delta}]$. Thus, since $V[G_{<\delta}]$ agrees with M well beyond the first inaccessible cardinal, φ holds also in M . Finally, since the real parameters of φ are fixed by j , it follows by elementarity that φ holds also in V , as desired. \square

In the previous argument, one can clearly relax the notion of “local” assertions quite a bit. Rather than using the least inaccessible cardinal, for example, one could use the least Mahlo cardinal or the second measurable cardinal or the fifth measurable limit of measurable cardinals, and so on. All that is required in the argument is that the cardinal in question be less than or equal to the Woodin cardinal δ used in the proof, and that the cardinal be the same in M and $V[G_{\delta}]$.

Finally, there is a seeming affinity between the Maximality Principles and the possibility that the theory of $L(R)$ is persistent, that is, that it is invariant by set forcing, as it is if there are sufficiently many Woodin cardinals. But what is the exact connection between these notions?

Question 26 *What is the relationship between $\square_{\mathcal{M}\mathcal{P}}$ and the $\text{Th}(L(\mathbb{R}))$ being invariant by forcing?*

5 Modified Maximality Principles

The Maximality Principle admits numerous modified forms, obtained by restricting the kinds of forcing extensions allowed in the modal quantifiers. Specifically, suppose that \mathcal{P} is a definable class of forcing notions; we may define that a statement φ is possible or forceable *by forcing notions in \mathcal{P}* , written $\Diamond_{\mathcal{P}} \varphi$, when it holds in a forcing extension $V^{\mathbb{P}}$ by a forcing notion in \mathcal{P} . Similarly, φ is necessary or persistent *by forcing notions in \mathcal{P}* , written $\Box_{\mathcal{P}} \varphi$, when φ holds in V and all extension $V^{\mathbb{P}}$ by forcing $\mathbb{P} \in \mathcal{P}$. Combining the two notions, we see that φ is forceably necessary by forcing notions in \mathcal{P} , written $\Diamond_{\mathcal{P}} \Box_{\mathcal{P}} \varphi$, when it holds in some extension $V^{\mathbb{P}}$ and all subsequent extensions $V^{\mathbb{P} * \dot{\mathbb{Q}}}$, where \mathbb{P} and $\dot{\mathbb{Q}}$ are taken from \mathcal{P} (interpreting \mathcal{P} , in the case of $\dot{\mathbb{Q}}$, in the model $V^{\mathbb{P}}$).

The Maximality Principle can now be stated in the restricted form as follows:

Maximality Principle ($\text{MP}_{\mathcal{P}}$) *If a statement is forceably necessary by forcing notions in \mathcal{P} , then it is persistent by forcing in \mathcal{P} .*

Natural instances of this principle include:

- (MP_{card}) Any statement forceably necessary by forcing that preserves cardinals is persistent by such forcing.
- (MP_{ω_1}) Any statement forceably necessary by forcing that preserves ω_1 is persistent by such forcing.
- (MP_{ccc}) Any statement forceably necessary by c.c.c. forcing is persistent by c.c.c. forcing.
- ($\text{MP}_{\text{proper}}$) Any statement forceably necessary by proper forcing is persistent by such forcing.
- ($\text{MP}_{\text{semi-proper}}$) Any statement forceably necessary by semi-proper forcing is persistent by such forcing.
- (MP_{CH}) Any statement forceably necessary by forcing preserving CH is persistent by such forcing.

And so on. All these principles admit the equivalent forms of Theorem 1, so that MP_{CH} , for example, is equivalent to the assertion that every true sentence is forceable by CH-preserving forcing over every forcing extension with CH.

But are they consistent? One might initially think, since these principles simply restrict the class of forcing notions in the Maximality Principle, that they are special cases of or at least follow immediately from the full Maximality Principle MP. But this is not correct. For the reader will readily realize that an assertion may be forceably necessary by c.c.c. forcing, for example, but not be forceably necessary at all (e.g. $\neg\text{CH}$). An assertion can be persistent under c.c.c. forcing, but not under all forcing.

So it is not immediately clear that these modified Maximality Principles are consistent. I will prove now that at least some of them are.

Theorem 27 *Suppose that \mathcal{P} is necessarily closed under finite support or countable support iterations of countable length. Assuming $V_\delta \prec V$, then, there is a forcing extension obtained by forcing with a notion in \mathcal{P} in which $\text{MP}_{\mathcal{P}}$ holds.*

Proof: In the proof of Lemma 5.5, let us simply ensure that the forcing \mathbb{Q}_n at stage n comes from \mathcal{P} . Then, define \mathbb{P} to be the finite support or countable support iteration of the \mathbb{Q}_n , depending on the closure of \mathcal{P} . Having done so, the iteration \mathbb{P} , as well as the tail forcing \mathbb{P}_{tail} , will be in \mathcal{P} , and so the argument of Lemma 5.5 works to establish $\text{MP}_{\mathcal{P}}$ in $V^{\mathbb{P}}$. \square

Corollary 28 *The principles MP_{ccc} , $\text{MP}_{\text{proper}}$ and $\text{MP}_{\text{semi-proper}}$ are equiconsistent with ZFC.*

Theorem 29 *Assume that ZFC is consistent. If in any model of set theory, the class \mathcal{P} is closed under finite iterations and members of \mathcal{P} are necessarily in \mathcal{P} , then $\text{ZFC} + \text{MP}_{\mathcal{P}}$ is consistent.*

Proof: This argument follows the first proof of Theorem 5. Suppose that M is any model of ZFC, and let T be the collection of sentences that are forceably necessary over M , referring throughout this argument only to forcing notions in \mathcal{P} .

I claim first that the theory T is consistent. Given any finite subcollection $\varphi_1, \dots, \varphi_n$, with φ_i forced necessary by \mathbb{P}_i , consider $\mathbb{P} = \mathbb{P}_1 \times \dots \times \mathbb{P}_n$. By the hypotheses of the theorem, this is in \mathcal{P} . Further, the extension $M^{\mathbb{P}}$ is a forcing extension of each $M^{\mathbb{P}_i}$, by forcing in \mathcal{P} there. Thus, the extension $M^{\mathbb{P}}$ satisfies each φ_i , and so this collection is consistent.

Next, I claim that T is absolute to all forcing extensions of M . If a sentence is forceably necessary over M with respect to forcing in \mathcal{P} , then the forcing that made it necessary will still be in \mathcal{P} in any forcing extension, and serve to make it necessary there. So it will be forceably necessary over any forcing extension also. Conversely, if a sentence is forceably necessary over a forcing extension of M by forcing in \mathcal{P} , then clearly it is forceably necessary in M .

Finally, I claim that any model N of T is a model of $\text{ZFC} + \text{MP}_{\mathcal{P}}$. Certainly $\text{ZFC} \subseteq T$ since these are necessary over M . If φ is forceably necessary over N , then φ must be in T , for if not, then the assertion that φ is not forceably necessary would be necessary and hence in T , a contradiction. Thus, $\varphi \in T$ and so φ holds in N , as desired. \square

Define that a poset is *necessarily-c.c.c.* if it is c.c.c. in every c.c.c. forcing extension. For example, while Souslin trees lose their c.c.c. property after forcing with them, the forcing to add (any number of) Cohen reals does not. Let $\text{MP}_{\square_{\text{ccc}}}$ be the corresponding Maximality Principle for such forcing notions. Similarly, define that \mathbb{P} is *necessarily ω_1 -preserving* if it preserves ω_1 over every forcing extension with the correct ω_1 , and denote the corresponding Maximality Principle by $\text{MP}_{\square_{\omega_1}}$.

Corollary 30 *The principles $\text{MP}_{\square_{\text{ccc}}}$ and $\text{MP}_{\square_{\omega_1}}$ are equiconsistent with ZFC.*

Proof: The corresponding classes of posets have the closure properties of the theorem. \square

Let me turn now to boldface versions of the modified principles. The intriguing possibility here is the potential to go beyond the reals by including some uncountable parameters in the scheme. In Observation 3, the argument that excluded uncountable parameters, we used forcing that collapsed cardinals. So when restricting to a class of posets that preserve all cardinals, such as the c.c.c. posets, we might hope to include many more parameters.

But unfortunately, we cannot hope to use any set as a parameter. In the c.c.c. case, for example, suppose that A is the set of reals of a model M . The assertion, “*there is a real $x \notin A$,*” is forceably necessary by c.c.c. forcing over M , because one can simply add a Cohen real, but it is not true in M by the very definition of A . More generally, if A is any set in M having hereditary size at least 2^ω , then the assertion, “ *2^ω is greater than the hereditary size of A ,*” is forceably necessary by c.c.c. forcing, since one may make 2^ω as large as desired, but again, it is not true in M .

This argument suggests that we must at least restrict our parameter space to $H(2^\omega)$, those parameters of hereditary size less than 2^ω . And the following theorem shows that restricting this far is enough.

Theorem 31 *The following theories are equiconsistent.*

1. $\text{ZFC} + \mathfrak{MP}_{ccc}$, allowing arbitrary parameters from $H(2^\omega)$, plus 2^ω is weakly inaccessible.
2. $\text{ZFC} + \mathfrak{MP}_{ccc}$, allowing ordinal parameters below 2^ω , plus 2^ω is regular.
3. $\text{ZFC} + V_\delta \prec V + \delta$ is inaccessible.
4. $\text{ZFC} + \mathfrak{MP}$.

Proof: Clearly, the first theory directly implies the second theory. Furthermore, I have already shown that the third and fourth theories are equiconsistent.

Lemma 31.1 *Every model of theory 2 contains an inner model of theory 3.*

This Lemma is a consequence of the following Lemma:

Lemma 31.2 *If \mathfrak{MP}_{ccc} holds (in the language with ordinal parameters below 2^ω) and $\delta = 2^\omega$ is regular, then δ is inaccessible in L and $L_\delta \prec L$.*

Proof: Suppose that the Maximality Principle \mathfrak{MP}_{ccc} holds for c.c.c. forcing in the language with ordinal parameters below $\delta = 2^\omega$, and that δ is regular. Note that for any $\gamma < \delta$, the assertion “ $2^\omega > \gamma^+$ ” is forceably necessary, since by c.c.c. forcing, one can pump up 2^ω as large as desired. Thus, this assertion must be true, and so δ is weakly inaccessible and hence inaccessible in L .

To prove $L_\delta \prec L$, I will simply verify the Tarski-Vaught criterion. Suppose that $L \models \exists x \psi(x, y)$ for some $y \in L_\delta$. Let $\beta < \delta$ be large enough so that $y \in L_\beta$. Now, by the Replacement Axiom, let α be least so that for every $y' \in L_\beta$ with an x satisfying $\psi(x, y)$, there is such an x in L_α . This property defines α in any forcing extension, and so the assertion $2^\omega > \alpha$ is forceably necessary. Thus, $\alpha < \delta$, and so the witness x for y can be found in L_δ , as desired. \square

Lemma 31.3 *Any model of theory 3 has a forcing extension that is a model of theory 1.*

Proof: This argument follows Lemma 10.2 above. Suppose that $V_\delta \prec V$ and δ is inaccessible. Enumerate $\langle \varphi_\alpha \mid \alpha < \delta \rangle$ all sentences φ (with unbounded repetition) in the language of set theory with (names for) parameters in V_δ coming from a forcing extension by forcing of size less than δ . Define a finite support δ -iteration of c.c.c. forcing, so that at each stage $\gamma < \delta$, the forcing \mathbb{Q}_α forces φ_α to be necessary over $V_\delta^{\mathbb{P}_\alpha}$, if possible (i.e. first, if this makes sense, if the parameters appearing in φ_α are \mathbb{P}_α -names, and second, if there is such a forcing in $V_\delta^{\mathbb{P}_\alpha}$). Let $G \subseteq \mathbb{P}_\delta$ be V -generic for \mathbb{P}_δ , and consider the model $V[G]$. Note that periodically during the iteration, the posets will ensure that 2^ω is made arbitrarily large below δ , and so $2^\omega \geq \delta$ in $V[G]$. Conversely, since the forcing has size δ , it is easy to see that $V[G] \models 2^\omega = \delta$. Further, since the iteration is c.c.c., δ remains a regular limit cardinal. Finally, if φ is a forceably necessary assertion over $V[G]$ with parameters in $H(2^\omega) = H(\delta)$, then $\varphi = \varphi_\alpha$ for some α such that the parameters are \mathbb{P}_α -names. Since φ was forceably necessary in $V[G]$ it must have been forceably necessary in $V[G_\alpha]$ and hence, since $V_\delta[G_\alpha] \prec V[G_\alpha]$, it was forceably necessary in $V_\delta[G_\alpha]$. Thus, it was forced to be necessary in $V_\delta[G_{\alpha+1}]$, and so it was necessary in $V[G_{\alpha+1}]$. Thus, it is true in $V[G]$, as desired. In summary, $V[G] \models \mathbb{MP}_{ccc} + 2^\omega$ is weakly inaccessible. \square

This completes the proof of the theorem. \square

A nearly identical argument works in the case of proper forcing:

Theorem 32 *The following theories are equiconsistent.*

1. $\text{ZFC} + \mathbb{MP}_{\text{proper}}$, using parameters in $H(2^\omega)$, plus 2^ω is weakly inaccessible.
2. $\text{ZFC} + \mathbb{MP}_{ccc}$, using parameters in $H(2^\omega)$, plus 2^ω is weakly inaccessible.
3. $\text{ZFC} + \mathbb{MP}$.

Proof: Using the techniques of 31.2, one can show that if the first theory holds, then $\delta = 2^\omega$ is inaccessible in L and $L_\delta \prec L$. And this has already been proved equiconsistent with the second and third theories.

Conversely, the second and third theories are equiconsistent with $\text{ZFC} + V_\delta \prec V$ for an inaccessible cardinal δ . Given this, one enumerates the formulas $\langle \varphi_\alpha \mid \alpha < \delta \rangle$ using (names

for) parameters in V_δ appearing in forcing extensions of size less than δ , and then performs a countable support δ -iteration \mathbb{P}_δ of proper forcing, which at stage $\alpha < \delta$ forces the necessity of φ_α over $V_\delta^{\mathbb{P}_\alpha}$ by proper forcing, provided that this make sense (that is, the names in φ_α are \mathbb{P}_α -names) and that this is possible (that is, there is a proper forcing poset accomplishing this). The resulting forcing extension $V^\mathbb{P}$ will be a model of $\text{ZFC} + \mathfrak{MP}_{\text{proper}}$, just as in Lemma 31.3. \square

And by using revised countable support in the iteration, one can deduce a similar fact for $\mathfrak{MP}_{\text{semi-proper}}$, using parameters in $H(2^\omega)$.

A natural question remains open here, whether the assertion that 2^ω is weakly inaccessible can be removed from Theorems 31 and 32. The arguments I gave show that $\mathfrak{MP}_{\text{ccc}}$ implies that 2^ω is a limit cardinal. And it is easy to see that it must be a limit cardinal of very high rank, since with c.c.c. forcing one can push 2^ω beyond \aleph_{ω_1} , $\aleph_{\aleph_{\omega_1}}$ or any other definable cardinal. Does $\mathfrak{MP}_{\text{ccc}}$ imply outright that 2^ω is regular? If so, then $\mathfrak{MP}_{\text{ccc}}$ is equiconsistent with \mathfrak{MP} . If not, then is $\mathfrak{MP}_{\text{ccc}}$ actually weaker than \mathfrak{MP} in consistency strength?

Another interesting point is that while the arguments here are based on that of Lemma 10.2, in that earlier lemma we could actually say much more about the nature of the δ -iteration \mathbb{P} , namely, that the forcing \mathbb{P} there was necessarily the Lévy collapse making δ the ω_1 of the extension. In the previous argument and in Lemma 31.3, this is certainly not the case. But is there, nevertheless, an alternative, simple characterization of the iterations \mathbb{P} ? If so, this would be very interesting to know.

There are truly a plethora of open problems remaining here. Apart from the big open question—the relative consistency of $\square\mathfrak{MP}$ —one can analyze the various modified maximality principles, including MP_{card} , MP_{ω_1} , $\text{MP}_{\text{stat-pres}}$, and so on, along with their boldface and necessary forms ($\mathfrak{MP}_{\text{card}}$, $\square\mathfrak{MP}_{\text{card}}$, etc.). Other interesting principles are obtained by restricting to forcing extensions that preserve a given statement or theory, such as CH. For example, MP_{CH} is the scheme asserting that every statement that is forceably necessary by forcing preserving CH is already true. One can also consider forcing notions not adding reals, or preserving a given supercompact cardinal, and so on. The possibilities seem endless.

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